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Elliptic genera on non-spin Riemannian symmetric spaces with $b_2 = 0$

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Abstract

We prove that the elliptic genera of the real Grassmannian $\mathbb{G}r_4(\mathbb{R}^{2m+5})$, $m \geq 1$, and the space $F_4/\mathrm{Sp}(3)\mathrm{Sp}(1)$ are identically zero. These vanishings are consistent with the rigidity under S^1 actions of the elliptic genera on these non-spin manifolds, and imply that their signatures are zero.

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1. Introduction

Hirzebruch and Slodowy [7] studied the elliptic genus on homogeneous spaces with an emphasis on those which are spin. In particular, they showed that the elliptic genus vanishes completely in many cases. There are, however, several non-spin homogeneous spaces whose elliptic genus has not been computed and which, as we shall see, also vanishes completely. They are the irreducible Riemannian symmetrics spaces with $b_2 = 0$.

The elliptic genus of a compact, oriented smooth $4n$-dimensional manifold M can be defined by the formal power series:

$$\tau_q(M) = \sum_{i=0}^{\infty} \tau(M, R_i) q^i,$$

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where $\tau(M, R_i)$ is the index of the signature operator with coefficients in a (virtual) vector bundle R_i. The bundles R_i are given by the following series:

$$R(q, T) = \sum_{i=0}^{\infty} R_i q^i = \bigotimes_{i=1}^{\infty} \bigwedge\nolimits_{q^i} T \otimes \bigotimes_{i=1}^{\infty} S_{q^i} T,$$

where $T = TM_{\mathrm{c}}$ is the complexified tangent bundle,

$$S_a T = \sum_{j=0}^{\infty} a^j S^j T, \qquad \bigwedge\nolimits_a T = \sum_{j=0}^{\infty} a^j \bigwedge\nolimits^j T$$

and $S^j T$, $\bigwedge^j T$ denote the jth symmetric and exterior tensor powers of T, respectively (cf. [6]). The first few terms of the sequence are

$$R_0 = 1, \qquad R_1 = 2T, \qquad R_2 = 2(T^{\otimes 2} + T),$$
$$R_3 = 2T^{\otimes 3} + \bigwedge\nolimits^3 T + S^3 T + 4T^{\otimes 2} + 2T.$$

Notice that the constant term of $\tau_q(M)$ is the signature of M, $\tau(M)$. This genus has been well studied on spin manifolds [3,6,9,10,12–14], for which Witten conjectured its rigidity under S^1 actions. The rigidity theorem was proved by Taubes [12] and Bott–Taubes [3].

By means of a change of coordinate in q, one obtains the alternative expression (cf. [6]):

$$\tilde{\tau}_q(M) = \frac{1}{q^{n/2}} \sum_{j=0}^{\infty} \hat{A}(M, R'_j) q^j,$$

where R'_j is the sequence of (virtual) bundles given by

$$R'(q, T) = \bigotimes_{k=2m+1} \bigwedge\nolimits_{-q^k} T \otimes \bigotimes_{k=2m+2} S_{q^k} T$$

and

$$\hat{A}(M, R'_j) = \langle \hat{A}(M) \cdot \mathrm{ch}(R'_j), [M] \rangle$$

is a twisted $\hat{A}$-genus. The first few terms of the sequence are

$$R'_0 = 1, \qquad R'_1 = -T, \qquad R'_2 = \bigwedge\nolimits^2 T + T, \qquad R'_3 = -\bigwedge\nolimits^3 T - T - T \otimes T.$$

If M is spin, $\tilde{\tau}_q(M)$ is a power series whose coefficients are indices of the Dirac operator with coefficients in the bundles R'_i, i.e. all the coefficients are integers.

We have found, however, that the rigidity of $\tau_q(M)$ under S^1 actions still holds on non-spin manifolds M with finite second homotopy group [4,5]. Furthermore, we have also proved the vanishing of the first coefficient of $\tilde{\tau}_q(M)$ on these manifolds

$$\hat{A}(M) = 0,$$

although it is no longer the index of the Dirac operator. Such a vanishing is similar to that on spin manifolds with S^1-actions [1], and is to be contrasted with the situation on non-spin manifolds with infinite second homotopy group as, for example:

$$\hat{A}(\mathbb{CP}^2) = -\tfrac{1}{8}.$$

Motivated by such a rigidity result, this paper is devoted to compute the entire elliptic genus of the non-spin real Grassmannians of 4-planes in $\mathbb{R}^{2m+5}$, $\mathbb{G}r_4(\mathbb{R}^{2m+5})$, $m \geq 1$, and the space $F_4/\mathrm{Sp}(3)\mathrm{Sp}(1)$.

Theorem 1.1. *The elliptic genera of* $\mathbb{G}r_4(\mathbb{R}^{2m+5})$, $m \geq 1$, *and of* $F_4/\mathrm{Sp}(3)\mathrm{Sp}(1)$ *vanish identically*:

$$\tau_q(\mathbb{G}r_4(\mathbb{R}^{2m+5})) = 0, \qquad \tilde{\tau}_q(\mathbb{G}r_4(\mathbb{R}^{2m+5})) = 0,$$
$$\tau_q\left(\frac{F_4}{\mathrm{Sp}(3)\mathrm{Sp}(1)}\right) = 0, \qquad \tilde{\tau}_q\left(\frac{F_4}{\mathrm{Sp}(3)\mathrm{Sp}(1)}\right) = 0.$$

This result deals with the elliptic genera of some of the spaces not treated by Hirzebruch and Slodowy in [7], and generalizes [2, Theorem 23.3(iii)]. We shall prove the theorem by computing the coefficients of $\tilde{\tau}_q$ via using twistor transform. They are twisted $\hat{A}$-genera (characteristic numbers), but are not the indices of twisted Dirac operators. The Grassmannian $\mathbb{G}r_4(\mathbb{R}^{2m+5})$, $m \geq 1$, and $F_4/\mathrm{Sp}(3)\mathrm{Sp}(1)$ are quaternion-Kähler manifolds, i.e. their holonomy is contained in a group $\mathrm{Sp}(n)\mathrm{Sp}(1)$ for appropriate values of n. We refer the reader to [11] for a survey on quaternion-Kähler geometry.

2. Quaternion-Kähler preliminaries

The holonomy group $\mathrm{Sp}(n)\mathrm{Sp}(1) = \mathrm{Sp}(n) \times_{\mathbb{Z}_2} \mathrm{Sp}(1) \subset \mathrm{SO}(4n)$ of a $4n$-dimensional quaternion-Kähler manifold M determines the following decomposition of the complexified tangent bundle:

$$TM_{\mathrm{c}} = E \otimes H,$$

where the fiber of the (locally defined) bundle E is isomorphic to the standard representation $\mathbb{C}^{2n}$ of $\mathrm{Sp}(n)$, and the fibre of the (locally defined) bundle H is isomorphic to the standard representation $\mathbb{C}^2$ of $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$.

Let us also denote the standard representations of $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)$ by E and H, respectively. The irreducible representations of $\mathrm{Sp}(1)$ are the symmetric tensor powers S^qH of H, with $\dim(S^qH) = q + 1$. The group $\mathrm{Sp}(n)$ leaves invariant a skew form $\omega \in \bigwedge^2 E$. The primitive subspace $\bigwedge_0^p E \subset \bigwedge^p E$ is the Hermitian complement of $\omega \wedge \bigwedge^{p-2} E \subset \bigwedge^p E$ and is an irreducible representation of $\mathrm{Sp}(n)$, $p \leq n$. Let us state the following lemma for future use.

Lemma 2.1. *Let F be a vector space and $U = \mathbb{C}^2$ the standard representation of* $\mathrm{SU}(2)$. *Then we have the following K-theoretic identities*

$$\bigwedge^j(F\otimes U)=\sum_{k=0}^{[j/2]}\bigwedge^{j-k}F\otimes\bigwedge^k F\otimes(S^{j-2k}U-S^{j-2k-2}U),$$

$$S^j(F\otimes U)=\sum_{k=0}^{[j/2]}S^{j-k}F\otimes S^k F\otimes(S^{j-2k}U-S^{j-2k-2}U).$$

The $\mathrm{Sp}(n)\mathrm{Sp}(1)$-module $R^{p,q}=\bigwedge_0^p E\otimes S^q H$ gives rise to a well defined bundle on M only if $p+q$ is even. Otherwise, the corresponding bundle will only be locally defined. The spin group $\mathrm{Spin}(4n)$ acts on $\mathbb{C}^{4n}$ via $\mathrm{SO}(4n)$. The spin representation Δ splits as the sum of two irreducible representations of equal dimension of $\mathrm{Spin}(4n)$, $\Delta=\Delta_+\oplus\Delta_-$. Since $\mathrm{Sp}(n)\mathrm{Sp}(1)\subset\mathrm{SO}(4n)$, the spin representation Δ decomposes further under $\mathrm{Sp}(n)\mathrm{Sp}(1)$

$$\Delta=\sum_{q=0}^{n}R^{n-q,q}.$$

This decomposition says that a quaternion-Kähler manifold admits a globally defined spin bundle if n is even. It also allows us to define the Dirac operator with coefficients in $P^p(E)\otimes S^q H$:

$$D(P^p(E)\otimes S^q H):\Gamma(\Delta_+\otimes P^p(E)\otimes S^q H)\to\Gamma(\Delta_-\otimes P^p(E)\otimes S^q H),$$

whenever $n+p+q$ is even, where $P^p(E)$ is a bundle of tensors on E of pure degree p. $D(P^p(E)\otimes S^q H)$ is an elliptic differential operator and its index is

$$\mathrm{ind}(D(P^p(E)\otimes S^q H))=\langle\hat{A}(M)\,\mathrm{ch}(P^p(E)\otimes S^q H),[M]\rangle,$$

where $\hat{A}(M)$ is the $\hat{A}$-genus of the manifold and ch( ) the Chern character of the corresponding bundle. Notice that the right hand side of this equation defines a characteristic number on M for all p and q, regardless of the parity of $n+p+q$. In fact, it defines a polynomial in p and q of degree less than or equal to $2n+1$ with characteristic numbers of M as coefficients. This observation will allow us to compute numbers such as $\hat{A}(M)$ even when this number is not the index of an elliptic operator.

2.1. The twistor space and twistor transform

The twistor space Z of a complete $4n$-dimensional quaternion-Kähler manifold M with positive scalar curvature can be constructed as the projectivization of the locally defined bundle $H\to M$, so that the fibre of $\pi:Z\to M$ is $\mathbb{CP}^1$. It is a Kähler manifold of complex dimension $2n+1$. Let L be the positive line bundle over Z which restricted to the fibers is $L|_{\pi^{-1}(x)}\cong\mathcal{O}(2)$. The Levi-Civita connection of M determines a horizontal holomorphic distribution $\mathcal{D}$ in $T^{1,0}Z$, which is a complex contact structure such that

$$0\to\mathcal{D}\to T^{1,0}Z\to L\to 0.$$

We also have the local C^∞-isomorphisms:

$$\mathcal{D} \cong \pi^* E \otimes L^{1/2}, \qquad \pi^* H \cong L^{1/2} \oplus L^{-1/2}.$$

Consider the holomorphic Euler characteristics:

$$\chi(Z, \mathcal{O}(L^k \otimes P^p(\mathcal{D}^*))) = \sum_{i=0}^{2n+1} (-1)^i \dim H^i(Z, \mathcal{O}(L^k \otimes P^p(\mathcal{D}^*))),$$

where $P^p(\mathcal{D}^*)$ is a bundle of tensors on $\mathcal{D}^*$ of degree p. The cohomological version of the twistor transform that we shall need is the following [11]:

$$\hat{A}(M, P^p(E) \otimes S^q H) = \chi(Z, \mathcal{O}(L^k \otimes P^p(\mathcal{D}^*))),$$

where $q = n + 2k - p$. In fact, we shall consider these expressions as polynomials in p and q (p and k resp.) whose coefficients are characteristic numbers of M (Z resp.). We can express this transform schematically as follows:

$$P^p(E) \otimes S^q H \mapsto P^p(\pi^* E) \otimes L^{q-n/2}.$$

2.2. Wolf spaces and Weyl dimension formula

In order to carry out the calculations, let us recall the description of the homogeneous quaternion-Kähler manifolds with positive scalar curvature [15]. Let G and $\mathfrak{g}$ be a compact simple Lie group and its Lie algebra, respectively. Let $\mathfrak{H} \subset \mathfrak{g}_c$ be a Cartan subalgebra and let R be the set of roots of $\mathfrak{g}$ and ρ the maximal root with respect to the order on $\mathfrak{H}$. Define the following subalgebras:

$$\mathfrak{K}_0 = \mathrm{span}(\rho) \oplus \mathfrak{g}_\rho \oplus \mathfrak{g}_{-\rho} \cong \mathfrak{sp}(1), \qquad \mathfrak{K}_1 = \mathfrak{H} \oplus \sum_{\langle\alpha,\rho\rangle=0} \mathfrak{g}_\alpha,$$

where $\langle , \rangle$ is the Killing form on $\mathfrak{g}_c$ and $\mathfrak{g}_\alpha \subset \mathfrak{g}_c$ is the weight space of the root $\alpha \in \mathfrak{H}$. Then $\mathfrak{K}_0 \oplus \mathfrak{K}_1$ is a parabolic subalgebra of $\mathfrak{g}_c$ and the corresponding real form $\mathfrak{K} = \mathfrak{g} \cap (\mathfrak{K}_0 \oplus \mathfrak{K}_1)$ is the Lie algebra of $K = K_1\,\mathrm{Sp}(1)$. Thus $M = G/K$ is a quaternion-Kähler symmetric space and $Z = G/(K_1 U(1))$ is the twistor space.

Let $R(\mathfrak{K}_1 \oplus \mathfrak{u}(1))$ be the set of roots of $K_1 U(1) \subset G$, R^+ be the set of positive roots of G with $R(\mathfrak{K}_1 \oplus \mathfrak{u}(1))$ generated by simple roots, $\delta = (1/2)\sum_{\alpha\in R^+} \alpha$, and W be the Weyl group of G. Let $V(\lambda)$ be an irreducible representation for $K_1 U(1)$ with highest weight $\lambda \in R(\mathfrak{K}_1 \oplus \mathfrak{u}(1))$ and $\mathbf{V}(\lambda)$ the corresponding homogeneous vector bundle on G/K. Then, by the Bott–Borel–Weil theorem and the Weyl dimension formula [8]:

$$\chi(Z, \mathcal{O}(\mathbf{V}(\lambda))) = \pm \dim V(\lambda) = \pm \prod_{\alpha\in R^+} \frac{\langle\alpha, \delta+\lambda\rangle}{\langle\alpha, \delta\rangle}.$$

We shall apply this formula to virtual bundles $L^k \otimes P^p(\mathcal{D}^*)$, as if they really existed globally for all k and p.

3. Proof of the theorem

3.1. The Grassmannian

Let M denote the real Grassmannian $\mathbb{G}r_4(\mathbb{R}^{2m+5})$ throughout this section.

First of all, notice that under $K_0 = \mathrm{SO}(2m+1)\mathrm{Sp}(1) \subset \mathrm{Sp}(2m+1)$, $E = W \otimes V$ where W is the fundamental $2m+1$-dimensional representation of $\mathrm{SO}(2m+1)$ and V the two-dimensional representation of $\mathrm{Sp}(1) = \mathrm{SU}(2)$, so that

$$T(\mathbb{G}r_4(\mathbb{R}^{2m+5}))_{\mathrm{c}} = (W \otimes V) \otimes H.$$

Let $\{e_i, i = 1, \dots, m+2\}$ denote the canonical basis of $\mathbb{R}^{m+2}$. We have the following:

$$\mathfrak{H} = \mathrm{span}(\{\alpha_j = e_j - e_{j+1}, j = 1, \dots, m+1\} \cup \{\alpha_{m+2} = e_{m+2}\}),$$
$$\rho = e_1 + e_2 = (1, 1, 0, \dots, 0), \qquad \delta = \tfrac{1}{2}(2m+3, 2m+1, \dots, 1),$$
$$\mathfrak{K}_1 \cong \mathfrak{so}(2m+1) \oplus \mathfrak{su}(2) \supset \mathrm{span}(\alpha_3, \dots, \alpha_m) \oplus \mathrm{span}(\alpha_1),$$

where the last linear span is the Cartan subalgebra of $\mathfrak{K}_1$.

The twistor space is

$$\frac{\mathrm{SO}(2m+5)}{\mathrm{SO}(2m+1) \times \mathrm{SU}(2) \times_{\mathbb{Z}_2} U(1)},$$

where $\mathrm{SU}(2) \times_{\mathbb{Z}_2} U(1) = U(2)$. The fundamental representation Q of $U(2)$ has dominant weight $(1, 0)$ so that $L = \det(Q)$ has weight $(1, 1)$, and the representation V of $\mathrm{SU}(2)$ corresponds to $V = Q \otimes L^{1/2}$ and has weight $(1, 0) + (-1/2, -1/2) = (1/2, -1/2)$. The standard representation W of $\mathrm{SO}(2m+1)$ has dominant weight $(1, 0, \dots, 0) \in \mathbb{R}^m$ which will be embedded in $\mathfrak{so}(2m+5)$ as e_3.

Note that any cohomological expression such as

$$\hat{A}(M), \qquad \hat{A}(M, T), \qquad \hat{A}(M, \textstyle\bigwedge^2 T), \qquad \hat{A}(M, S^2 T)$$

can be transformed by twistor transform into a sum of (virtual) holomorphic Euler characteristics on the twistor space.

The trivial bundle gets transformed as follows:

$$1 \mapsto L^{-(2m+1)/2},$$

whose weight is $\gamma_0 = ((-2m-1)/2)(e_1+e_2)$. By adding δ we see that $\gamma_0+\delta = (1, 0, (2m-1)/2, \dots, 1/2)$ so that when contracted with the root $e_2 \in R^+$, we get 0. This proves that the characteristic number $\hat{A}(M) = 0$.

The tangent bundle gets transformed as follows:

$$E \otimes H \mapsto E \otimes L^{-m} \cong W \otimes V \otimes L^{-m},$$

whose weight is $\gamma_1 = (1/2)(e_1 - e_2) - m(e_1 + e_2) + e_3$. By adding δ we see that $\gamma_1 + \delta = (2, 0, (2m+1)/2, (2m-3)/2, \dots, 1/2)$ so that when contracted with the root e_2 we get zero, which proves that the characteristic number is $\hat{A}(M, T) = 0$. As we can see, the only two entries that matter are the first two.

The second exterior power of the tangent bundle is

$$\bigwedge^2 (E \otimes H) = \bigwedge^2 E \otimes (S^2 H - 1) + E \otimes E,$$

which gets transformed on the twistor space as

$$\bigwedge^2 E \otimes L^{1-2m/2} - \bigwedge^2 E \otimes L^{-2m-1/2} + E \otimes E \otimes L^{-2m-1/2}.$$

After another application of Lemma 2.1 it becomes

$$\bigwedge^2 W \otimes (S^2 V \otimes L^{(1-2m)/2} - L^{(1-2m)/2} - S^2 V \otimes L^{(-2m-1)/2} - L^{(-2m-1)/2}) + W \otimes W \otimes (L^{(1-2m)/2} + 2L^{(-2m-1)/2} + S^2 V \otimes L^{(-2m-1/2)}).$$

As we observed before, the only entries that matter are the first two, so we shall concentrate on them. Consider first the line of the factor $\bigwedge^2 W$; the relevant weights plus δ are

$$+(3, 0), \qquad -(2, 1), \qquad -(2, -1), \qquad +(1, 0),$$

the first and last ones produce zero, while the two middle ones have the same sign and are related by a reflection contained in the Weyl group of $\mathfrak{so}(2m + 5)$; they cancel each other out and produce zero too. Now consider the second line with factor $W \otimes W$; the relevant weights plus δ are

$$+(2, 1), \qquad +(1, 0), \qquad +(2, -1).$$

All of them produce zero, which proves $\hat{A}\left(M, \bigwedge^2 T\right) = 0$.

Clearly, for every weight γ that appears in the decomposition, either the second coordinate of $\gamma+\delta$ will be zero, or there will be another weight which will cancel it out. More precisely, since the decomposition of the tangent space $TM_{\mathrm{c}} = W \otimes V \otimes H$ is symmetrical in both SU(2) bundles V and H, for every summand in the virtual decomposition containing a factor $S^i V \otimes S^j H$ there will be a symmetrical one containing $S^j V \otimes S^i H$. Thus, if $P^l(W)$ represents a tensor bundle on W of degree l, the bundle $P^l(W) \otimes S^i V \otimes S^j H$ is transformed into $P^l(W) \otimes S^i V \otimes L^{j-(2m+1)/2}$, which gives the weight $((i + j - (2m + 1))/2, (j - i - (2m + 1))/2)$. By adding δ to it we get $((i + j + 2)/2, (j - i)/2)$. Similarly, the bundle $P^l(W) \otimes S^j V \otimes S^i H$ gets transformed into $P^l(W) \otimes S^j V \otimes L^{i-(2m+1)/2}$ with weight $((i+j-(2m+1))/2, (i-j-(2m+1))/2)$. By adding δ we get $((i+j+2)/2, (i-j)/2)$. Hence, either $i = j$ so that the second entry is zero, or the two resulting vectors are related by a reflection in the Weyl group and their corresponding values from the Weyl dimension formula cancel each other out.

In this fashion, the entire elliptic genus $\tilde{\tau}_q(\mathbb{G}r_4(\mathbb{R}^{2m+5}))$ vanishes, and so does $\tau_q(\mathbb{G}r_4(\mathbb{R}^{2m+5}))$. In particular, its signature vanishes:

$$\tau(\mathbb{G}r_4(\mathbb{R}^{2m+5})) = 0.$$

3.2. The exceptional space $F_4/Sp(3)Sp(1)$

The space $F_4/\mathrm{Sp}(3)\mathrm{Sp}(1)$ is also a 28-dimensional quaternion-Kähler manifold. Thus, we shall apply the same ideas. Let $M = F_4/\mathrm{Sp}(3)\mathrm{Sp}(1)$ throughout this section, and Z its twistor space.

The tangent space also splits as $E \otimes H$ under the holonomy group Sp(7)Sp(1), and decomposes further under Sp(3)Sp(1) which also acts irreducibly on the tangent space:

$$E = \bigwedge\nolimits_0^3 \tilde{E},$$

where $\tilde{E} = \mathbb{C}^6$ is the standard representation of Sp(3). Let $\mathfrak{H}$ be the Cartan subalgebra of $\mathrm{Lie}(F_4)_{\mathrm{c}}$ spanned by the following basic roots:

$$\{\alpha_1 = (1, -1, 0, 0), \qquad \alpha_2 = (0, 1, -1, 0),$$
$$\alpha_3 = (0, 0, 2, 0), \qquad \alpha_4 = (-1, -1, -1, 1)\}.$$

The coordinates have been chosen so that $\mathfrak{K}_1 = \mathfrak{sp}(3)$ with Cartan subalgebra spanned by $\{\alpha_1, \alpha_2, \alpha_3\}$, and which is orthogonal to the maximal root $\rho = (0, 0, 0, 2)$. In this case $\delta = (3, 2, 1, 8)$. The roots coming from Sp(3) will be embedded canonically in the first three coordinates and the one coming from $U(1)$ in the last coordinate.

Because of the modular properties of the elliptic genus [6] we only need to compute a few coefficients, namely:

$$\hat{A}(M), \qquad \hat{A}(M, -T), \qquad \hat{A}\left(M, \bigwedge\nolimits^2 T + T\right), \qquad \hat{A}\left(M, -\bigwedge\nolimits^3 T - T - T \otimes T\right).$$

Just as before

$$1 = 1, \qquad T = E \otimes H, \qquad \bigwedge\nolimits^2 T = \bigwedge\nolimits^2 E \otimes (S^2 H - 1) + E \otimes E,$$
$$\bigwedge\nolimits^3 T = \bigwedge\nolimits^3 E \otimes (S^3 H - H) + \bigwedge\nolimits^2 E \otimes E \otimes H,$$
$$T \otimes T = \left(\bigwedge\nolimits^2 E + S^2 E\right) \otimes (S^2 H + 1)$$

so that they get transformed to the twistor space as follows:

$$1 \mapsto L^{-7/2}, \qquad T \mapsto E \otimes L^{-3},$$
$$\bigwedge\nolimits^2 T \mapsto \bigwedge\nolimits^2 E \otimes (L^{-5/2} - L^{-7/2}) + E \otimes E \otimes L^{-7/2},$$
$$\bigwedge\nolimits^3 T \mapsto \bigwedge\nolimits^3 E \otimes (L^{-2} - L^{-3}) + \bigwedge\nolimits^2 E \otimes E \otimes L^{-3},$$
$$T \otimes T \mapsto \left(\bigwedge\nolimits^2 E + S^2 E\right) \otimes (L^{-5/2} + L^{-7/2}).$$

We need to identify the weight decomposition of the tensor powers of E in the representation ring of Sp(3):

$$E = V(1,1,1), \qquad \bigwedge^2 E = V(0,0,0) \oplus V(2,2,0),$$
$$\bigwedge^3 E = V(1,1,1) \oplus V(3,2,0),$$
$$E \otimes E = V(0,0,0) \oplus V(2,2,0) \oplus V(2,0,0) \oplus V(2,2,2),$$
$$S^2 E = V(2,0,0) \oplus V(2,2,2),$$
$$\bigwedge^2 E \otimes E = V(1,1,1) \oplus V(3,3,1) \oplus V(2,1,0) \oplus V(3,2,0) \oplus V(3,1,1).$$

Consider $L^{-7/2}$, which corresponds to $-7/2(0,0,0,2)$. When adding δ we get $(3,2,1,1)$, which is orthogonal to the root $(0,0,-1,1)$ and, therefore, produces zero in the dimension formula. Hence $\hat{A}(M) = 0$.

Consider $E \otimes L^{-3}$, which corresponds to $(1,1,1,-6)$. After adding δ we get $(4,3,2,2)$ which is again orthogonal to $(0,0,-1,1)$ and produces zero. Hence $\hat{A}(M,T) = 0$.

The rest of the calculations are analogous and most of the weights lead to zero, while just a couple cancel each other out. This proves that the elliptic genus vanishes $\tilde{\tau}_q(M) = 0$, and $\tau_q(M) = 0$ as well as the signature:

$$\tau\left(\frac{F_4}{\mathrm{Sp}(3)\mathrm{Sp}(1)}\right) = 0.$$

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